# Finite of Sense and Infinite of Thought

## A History of Computation, Logic, and Algebra

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Oracle Corporation

# Finite of Sense and Infinite of Thought

## A History of Computation, Logic, and Algebra



**https://pron.github.io/computation-logic-algebra**

- I will read out loud the highlighted bits

- Either the well was very deep, or she fell very slowly, for she had plenty of time, as she went down, to look about her. First, she tried to make out what she was coming to, but it was too dark to see anything; then she looked at the sides of the well and noticed that they were filled with cupboards and book-shelves; here and there she saw maps and pictures hung upon pegs. She took down a jar from one of the shelves as she passed. It was labeled "ORANGE MARMALADE," but, to her great disappointment, it was empty; she did not like to drop the jar, so managed to put it into one of the cupboards as she fell past it.

We must first state the subject of our inquiry and the faculty to which it belongs: its subject is demonstration and the faculty that carries it out demonstrative science. We must next define a premiss, a term, and a syllogism, …; and after that, the inclusion or noninclusion of one term in another

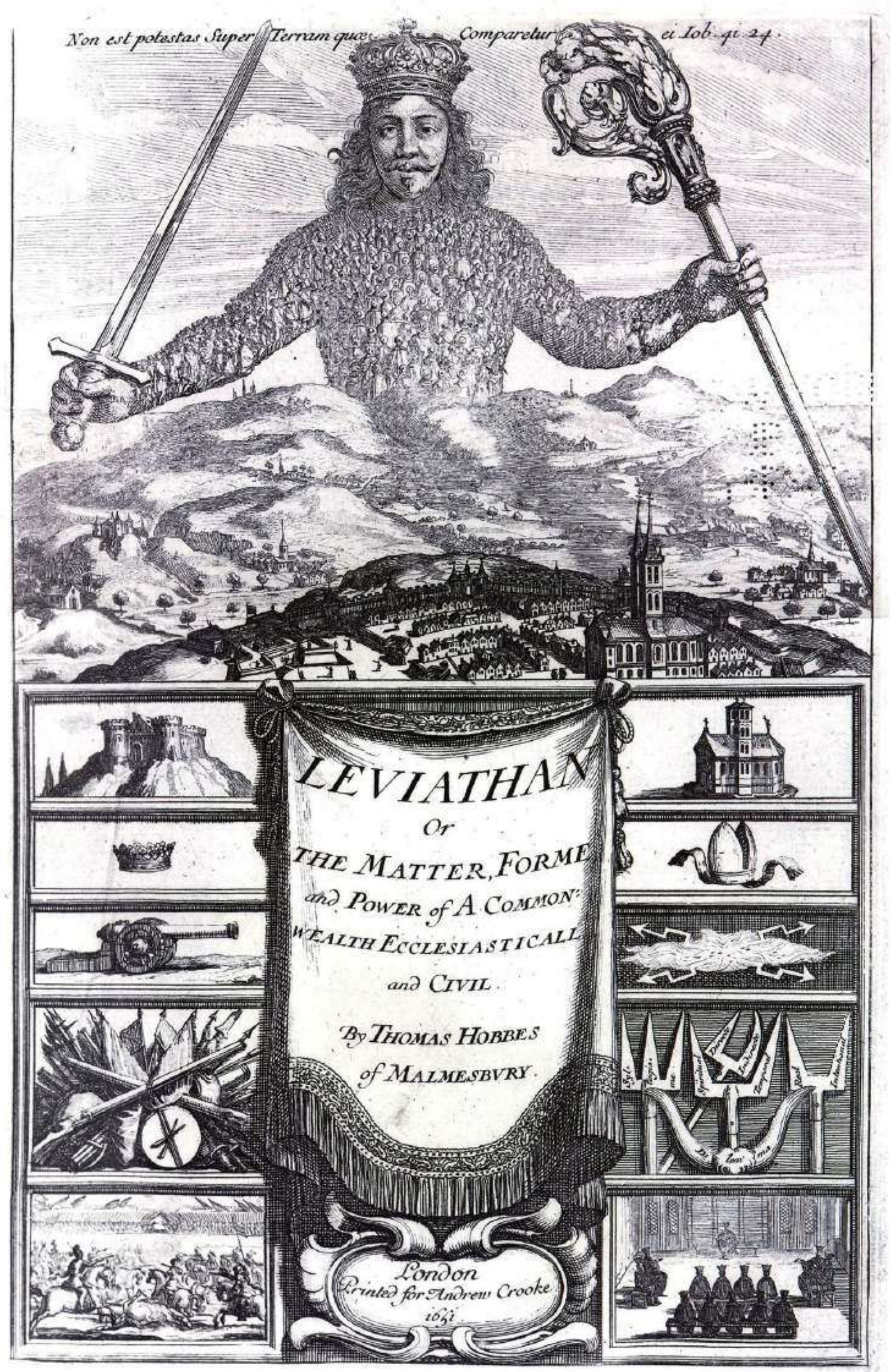
… A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.

… First then take a universal negative with the terms $A$ and $B$. If no $B$ is $A$, neither can any $A$ be $B$. For if some $A$ (say $C$) were $B$, it would not be true that no $B$ is $A$; for $C$ is a $B$.


— *Aristotle, Prior Analytics,* c. 350 BCE

**This logical truth depends upon the structure of the sentence and not upon the particular matters spoken of.**

— *Augustus De Morgan, Formal Logic,* 1847




Nature … is by the art of man… imitated, that it can make an Artificial Animal. For seeing life is but a motion of Limbs… ; why may we not say, that all Automata (engines that move themselves by springs and wheeles as doth a watch) … have an artificial life? For what is the Heart, but a Spring; and the Nerves, but so many Strings; and the Joynts, but so many Wheeles, giving motion to the whole Body, such as was intended by the Artificer?

— *Thomas Hobbes, Leviathan,* 1651

By RATIOCINATION, I mean *computation*. Now to compute, is either to collect the sum of many things that are added together, or to know what remains when one thing is taken out of another. Ratiocination therefore is the same with *Addition* and *Substraction*;

… We must not therefore thinke that Computation… has place onely in numbers…; for Magnitude, Body, Motion, Time, Degrees of Quality, Action, Conception, Proportion, Speech and Names (in which all the kinds of Philosophy consist) are capable of Addition and Substraction.

— *Thomas Hobbes, Computation or Logique,* 1656

Thomas Hobbes, everywhere a profound examiner of principles, rightly stated that everything done by our mind is a *computation*

— *Gottfried Wilhelm Leibniz, Dissertation On the Art of Combination,* 1666

[S]ince it is the nature of the soul to represent the universe … the sequence of representations which the soul produces will correspond naturally to the sequence of changes in the universe itself.

— *Gottfried Wilhelm Leibniz,* 1695

[O]ur thoughts are for the most part what I call 'blind thoughts'. I mean that they are empty of perception and sensibility, and consist in the wholly unaided use of symbols… Usually words are in this respect like the symbols of arithmetic and algebra. We often reason in words, with the object itself virtually absent from our mind.

[Language] serves also for purposes other than communication; for it also enables man to reason to himself, both because words provide the means for remembering abstract thoughts and also because symbols and 'blind thoughts' are useful in reasoning, as it would take too long to lay everything out and always replace terms by definitions.

— *Gottfried Wilhelm Leibniz,* 1704

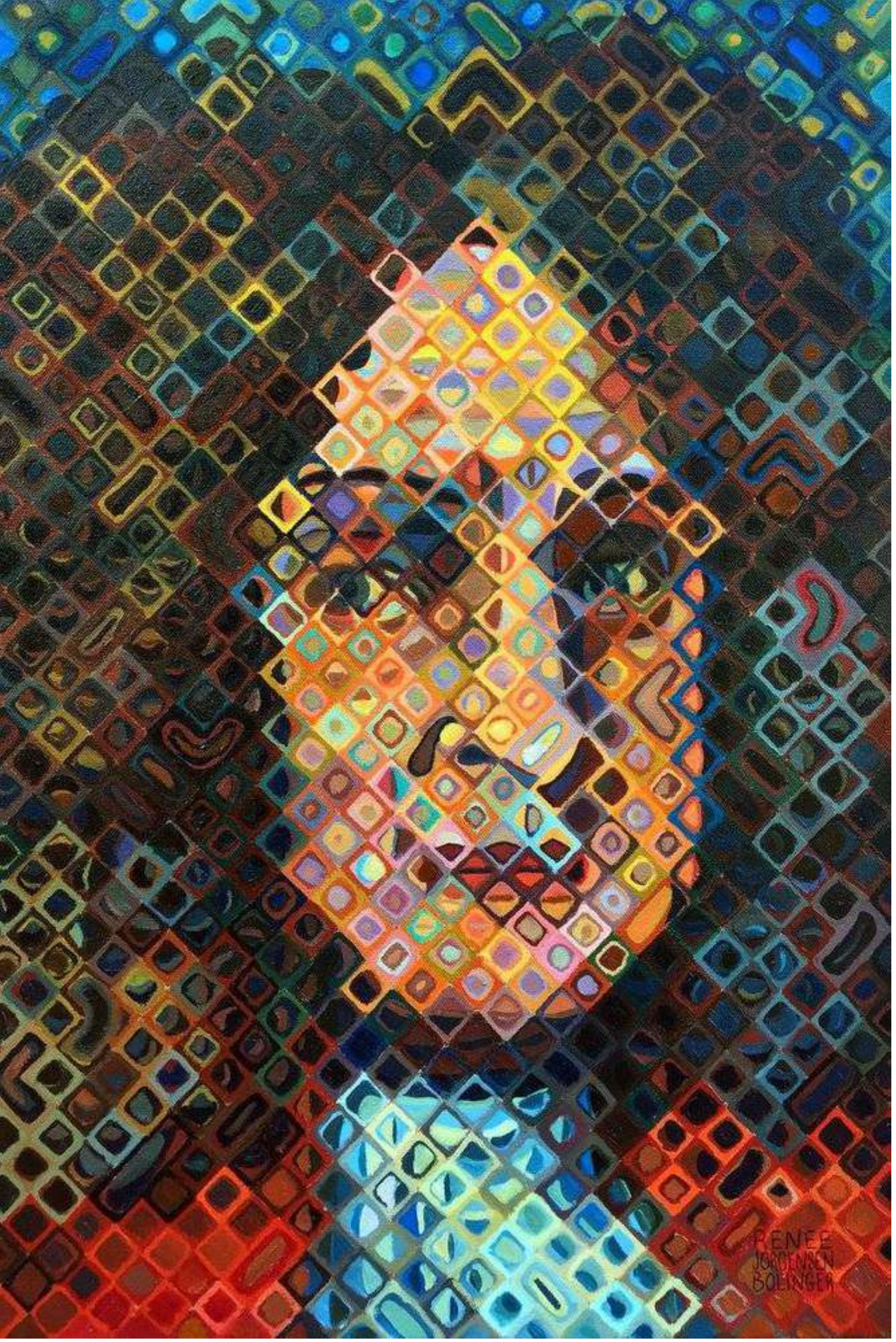
[A] calculus is nothing but operation through characters, and this has its place not only in matters of quantity but in all other reasoning as well. Meanwhile I have a very high regard for such problems as can be solved by mental powers alone insofar as this is possible, without a prolonged calculation, that is, without paper and pen. For such problems depend as little as possible on external circumstances, being within the power even of a captive who is denied a pen and whose hands are tied. Therefore we ought to practice both in calculating and in meditating…

— *Gottfried Wilhelm Leibniz,* 1678

# The Art of Combinations (Arte Combinatoria)
## Characteristica Universalis
## Calculus Ratiocinator

[T]he art of combinations in particular, as I take it (it can also be called a general characteristic or algebra), is that science in which are treated the forms or formulas of things in general, that is, quality in general or similarity and dissimilarity; in the same way that ever new formulas arise from the elements a, b, c themselves when combined with each other, whether these elements represent quantities or something else. This art is distinct from common algebra, which deals with formulas applied to quantity only or to equality and inequality.

— *Gottfried Wilhelm Leibniz, On Universal Synthesis and Analysis,* 1679



*Portrait by Renee Bolinger*

[W]hy should there not … be categories for complex terms, by which truths may be ordered? … It seemed to me… that this could be achieved universally if we first had the true categories for simple terms and if, to obtain these, we set up something new in the nature of an alphabet of thoughts…

… All derivative concepts, moreover, arise from a combination of primitive ones, and the more composite concepts from the combination of less composite ones.

— *Gottfried Wilhelm Leibniz, On Universal Synthesis and Analysis,* 1679

- *a* is *a*
- *ab* is *a*
- *a* is not non-*a*.
- Non-*a* is not *a*
- What is not *a* is non-*a*
- What is not non-*a* is *a*
- The repetition of the same letter in the same term is useless; thus [if *b* is *a* then] *b* is *aa*, or *bb* is *a*

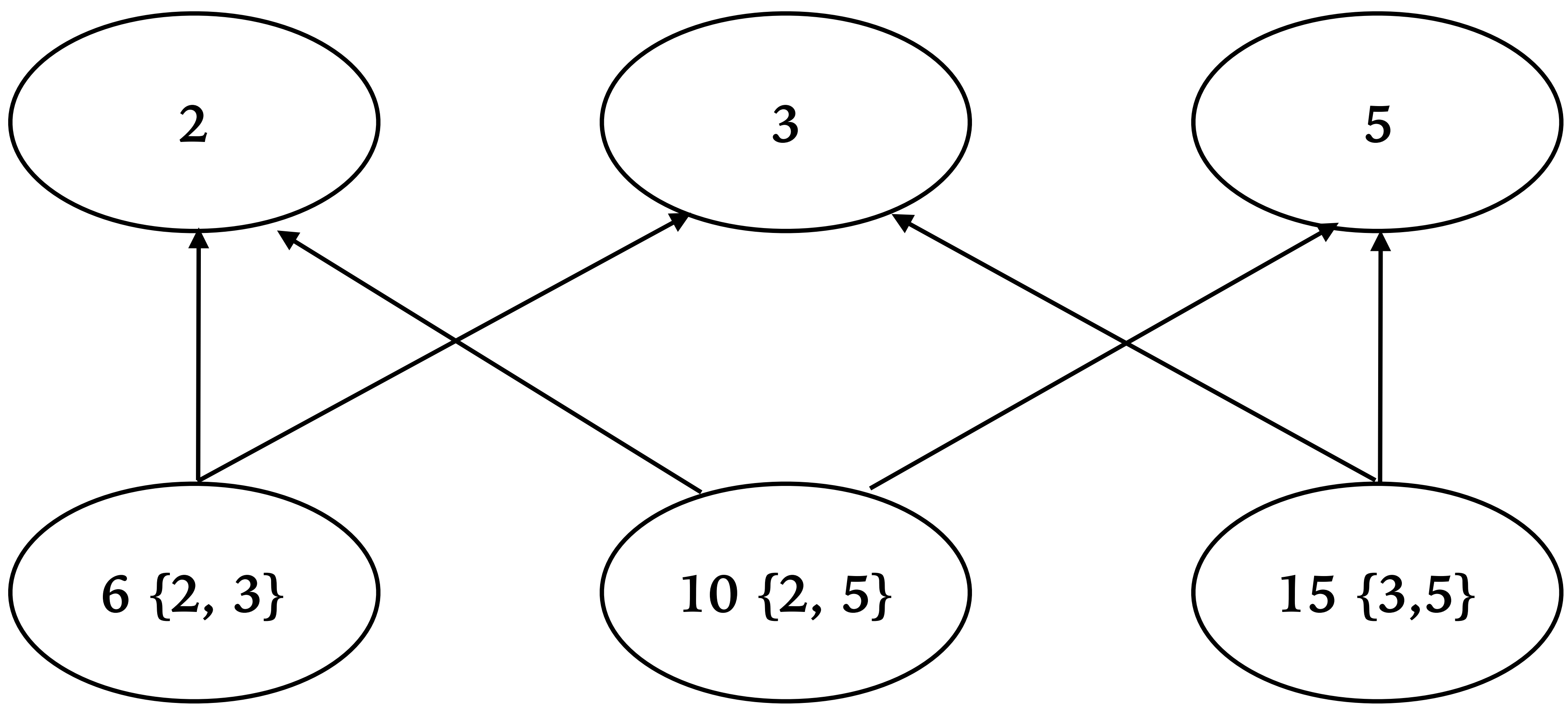
— *Gottfried Wilhelm Leibniz, Two Studies in the Logical Calculus,* 1679

- To every term whatever may be assigned its characteristic number

- *The rule for discovering fitting characteristic numbers* is this one only: when the concept of a given term is composed directly out of the concepts of two or more other terms, then the characteristic number of the given term is to be produced by multiplying the characteristic numbers of the terms composing it.

- Hence we can also determine through characteristic numbers which term does not contain another. One has merely to test whether the number of one term can be divided exactly by the number of the other.

— *Gottfried Wilhelm Leibniz, Two Studies in the Logical Calculus,* 1679


2
3
5
6 {2, 3}
10 {2, 5}
15 {3,5}

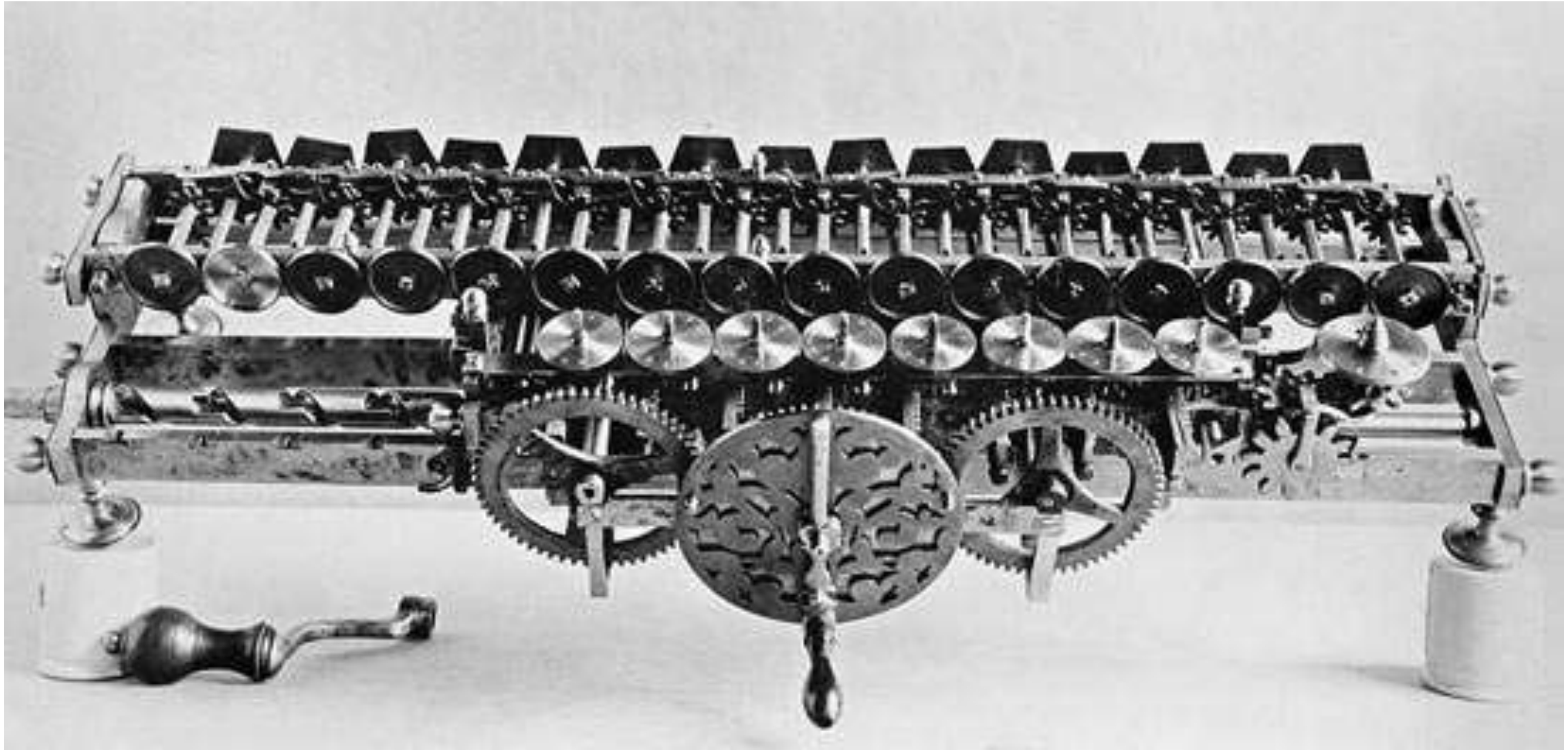
Once the characteristic numbers for most concepts have been set up… the human race will have a new kind of instrument which will increase the power of the mind much more than optical lenses strengthen the eyes and which will be as far superior to microscopes or telescopes as reason is superior to sight.

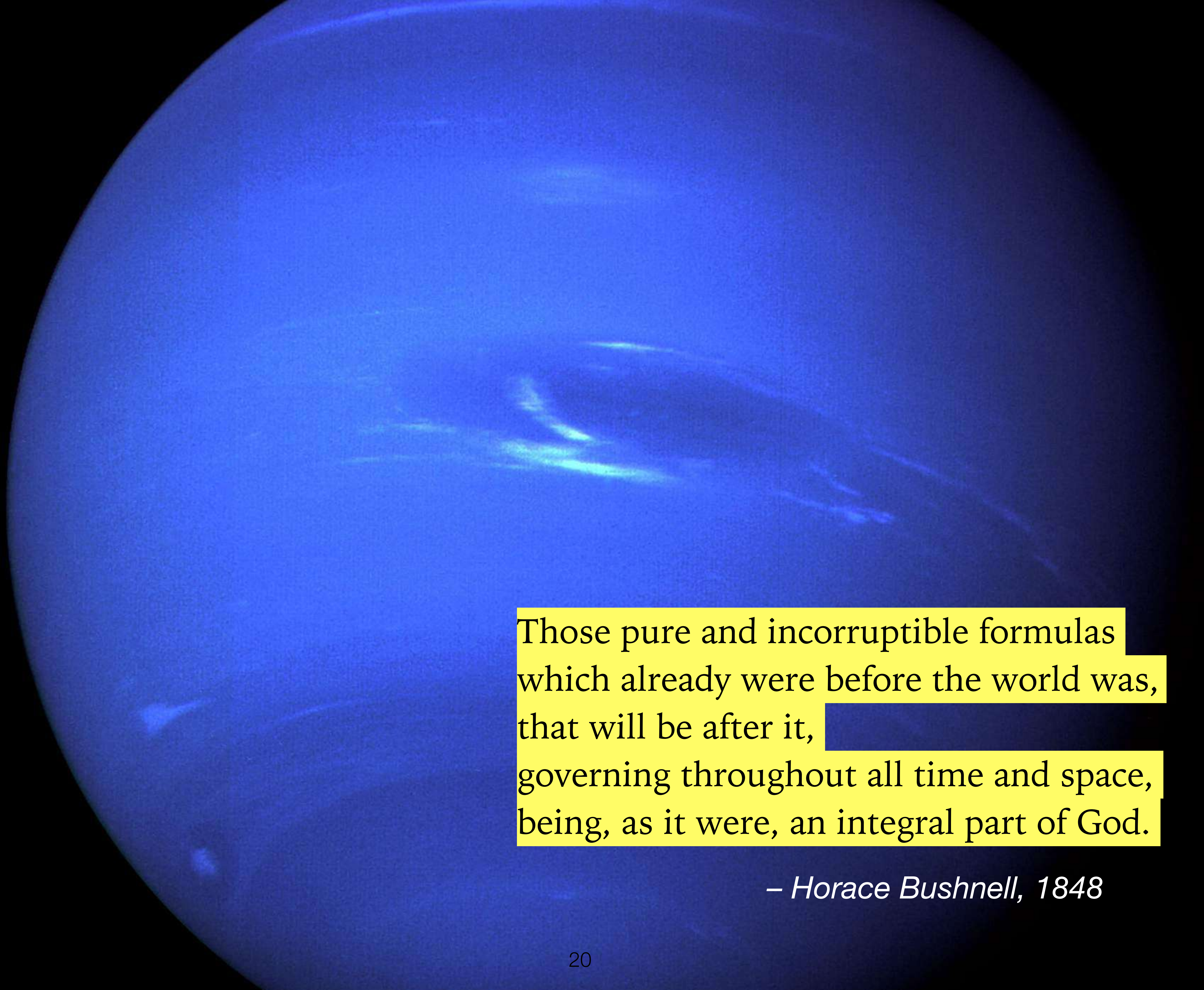
… [A]nyone who is certainly convinced of the truth of religion and its consequences, and … desires the conversion of mankind, will surely admit… that nothing will be more influential than this discovery for the propagation of the faith, unless it be miracles, the holiness of an apostle, or the victories of a great monarch.

*— Gottfried Wilhelm Leibniz, On the General Characteristic, 1679*

Leibniz und Sophie Charlotte von Preußen. (S. 140)

When this is done, if controversies were to arise, there would be be no more need of disputation between two philosophers than between two calculators. For it would suffice for them to take their pencils in their hands and to sit down at the abacus, and say to each other… : Let us calculate [calculemus].





Those pure and incorruptible formulas
which already were before the world was,
that will be after it,
governing throughout all time and space,
being, as it were, an integral part of God.

*– Horace Bushnell, 1848*

… [It is] necessary to consider symbols not merely as the general representatives of numbers, but of every species of quantity, and likewise to give a form to the definitions of the operations of Algebra, which must render them independent of any subordinate science: for in the first place the symbols, whatever they denote, must be unlimited in value, and it is only by their ceasing to be abstract numbers that we shall be enabled to interpret the *affections* which the signs + or – (or any other signs) essentially attached to them may be supposed to express

… If we should rest satisfied with such assumed rules for the combinations of symbols and of signs by such operations, which are perfectly independent of any interpretation of their meaning, or of their relation to each other, we should retain in the results obtained all the symbols which were incorporated

*– George Peacock, A Treatise on Algebra, 1830*

This application of an almost metaphysical system of abstract signs, by which the motion of the hand performs the office of the mind.

… The idea of calculation by mechanism is not new. … [O]ne of the most remarkable attempts of this kind which has been made since that of Pascal, was a machine invented by Leibnitz, of which we are not aware that any detailed or intelligible description was ever published. Leibnitz described its mode of operation, and its results, in the "Berlin Miscellany," but he appears to have declined any description of its details…

*–Dionysius Lardner, Babbage's Calculating Engine, 1834*

It will be easily admitted, that an assembly of eminent naturalists and physicians, with a sprinkling of astronomers, and one or two abstract mathematicians, were not precisely the persons best qualified to appreciate such an instrument of mechanical investigation as we have here described.

… We trust that a more auspicious period is at hand; that the chasm which has separated practical from scientific men will speedily close; and that that combination of knowledge will be effected, which can only be obtained when we see the men of science more frequently extending their observant eye over the wonders of our factories, and our great practical manufacturers, with a reciprocal ambition, presenting themselves as active and useful members of our scientific associations.

*–Dionysius Lardner, Babbage's Calculating Engine, 1834*

It is impossible to construct machinery occupying unlimited space ; but it is possible to construct finite machinery, and to use it through unlimited time. It is this substitution of the *infinity of time* for the *infinity of space* which I have made use of, to limit the size of the engine and yet to retain its unlimited power.

… Thus it appears that the whole of the conditions which enable a finite machine to make calculations of unlimited extent are fulfilled in the Analytical Engine. The means I have adopted are uniform. I have converted the infinity of space, which was required by the conditions of the problem, into the infinity of time. The means I have employed are in daily use in the art of weaving patterns.

*–Charles Babbage, Passages from the Life of a Philosopher, 1864*

On the first part of my inquiry I soon arrived at a demonstration that every game of skill is susceptible of being played by an automaton.... Hitherto I had considered only the philosophical view of the subject, but a new idea now entered my head which seemed to offer some chance of enabling me to acquire the funds necessary to complete the Analytical Engine.

It occurred to me that if an automaton were made to play this game, it might be surrounded with such attractive circumstances that a very popular and profitable exhibition might be produced. I imagined that the machine might consist of the figures of two children playing against each other, accompanied by a lamb and a cock. That the child who won the game might clap his hands whilst the cock was crowing, after which, that the child who was beaten might cry and wring his hands whilst the lamb began bleating.

*–Charles Babbage, Passages from the Life of a Philosopher, 1864*

That which renders Logic possible, is the existence in our minds of general notions, our ability to conceive of a class, and to designate its individual members by a common name. The theory of Logic is thus intimately connected with that of Language. A successful attempt to express logical propositions by symbols, the laws of whose combinations should be founded upon the laws of the mental processes which they represent, would, so far, be a step toward a philosophical language…

*–George Boole, The Mathematical Analysis of Logic, 1847*

1. Meaning – notion/class

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3. Computation – calculus of terms

*–George Boole, The Mathematical Analysis of Logic, 1847*

The design of the following treatise is to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of a Calculus, and upon this foundation to establish the science of Logic and construct its method;

… To deduce the laws of the symbols of Logic from a consideration of those operations of the mind which are implied in the strict use of language as an instrument of reasoning.

*–George Boole, An Investigation of the Laws of Thought, 1854*

PROPOSITION I
All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.:

1st. Literal symbols, as x, y, &c., representing things as subjects of our conceptions.

2nd. Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements.

3rd. The sign of identity, $=$.

*–George Boole, An Investigation of the Laws of Thought, 1854*

They who are acquainted with the present state of the theory of Symbolical Algebra, are aware, that the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible… This principle is indeed of fundamental importance

*–George Boole, The Mathematical Analysis of Logic, 1847*

1 — The “universe of discourse”
0 — The empty class
× — the mental operation of selection (intersection)
+ — union of *disjoint* classes (i.e. x + y is defined only if xy = 0)

“Boole’s equation” $x^2 = x$ for any x $x(1 - x) = 0$

All x is y $xy = x$

Some x is y $xy \neq 0$

PEIRCE: *On the Algebra of Logic.* 33

If $a \prec x$ and $b \prec x$,
then $a + b \prec x$;

and conversely,

if $a + b \prec x$,
then $a \prec x$ and $b \prec x$.

If $x \prec a$ and $x \prec b$,
then $x \prec a \times b$; (2)

and conversely,

if $x \prec a \times b$,
then $x \prec a$ and $x \prec b$. (3)

*–Charles Sanders Peirce, On the Algebra of Logic, 1880*

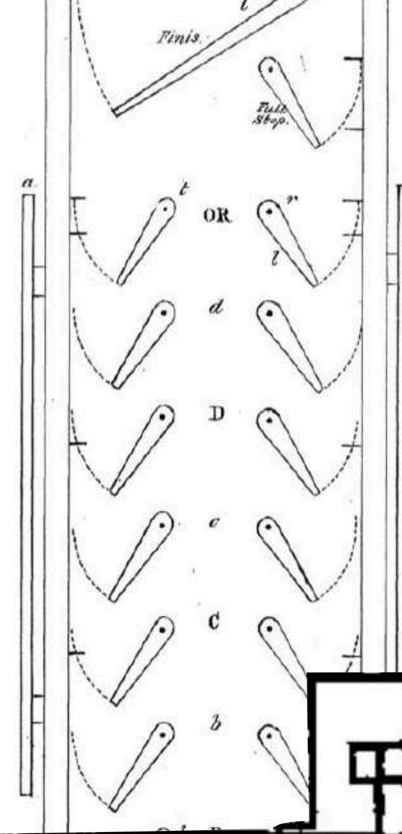
In order to gain a clear understanding of the origin of the various signs used in logical algebra and the reasons of the fundamental formulæ, we ought to begin by considering how logic itself arises.

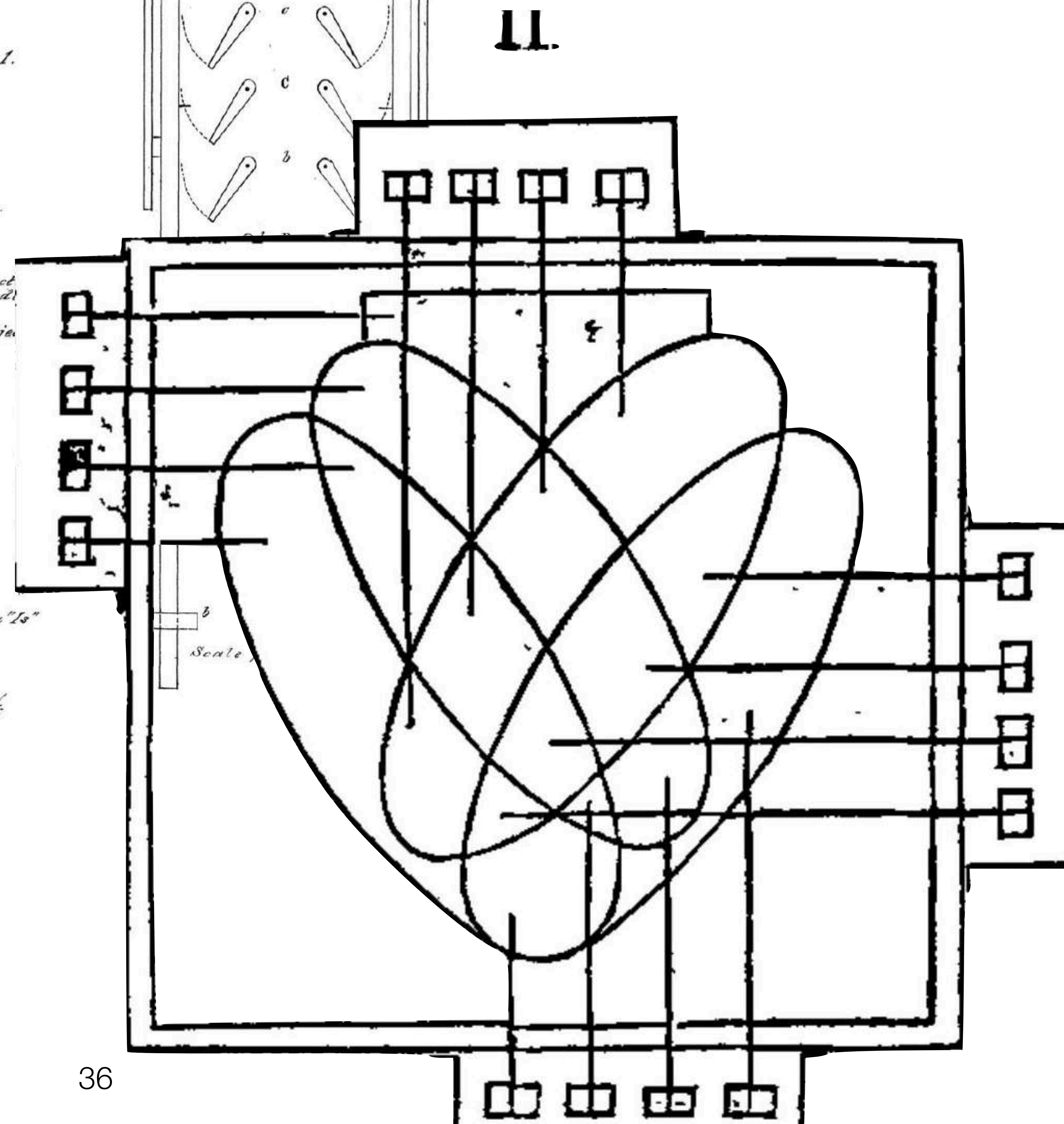
Thinking, as cerebration, is no doubt subject to the general laws of nervous action.

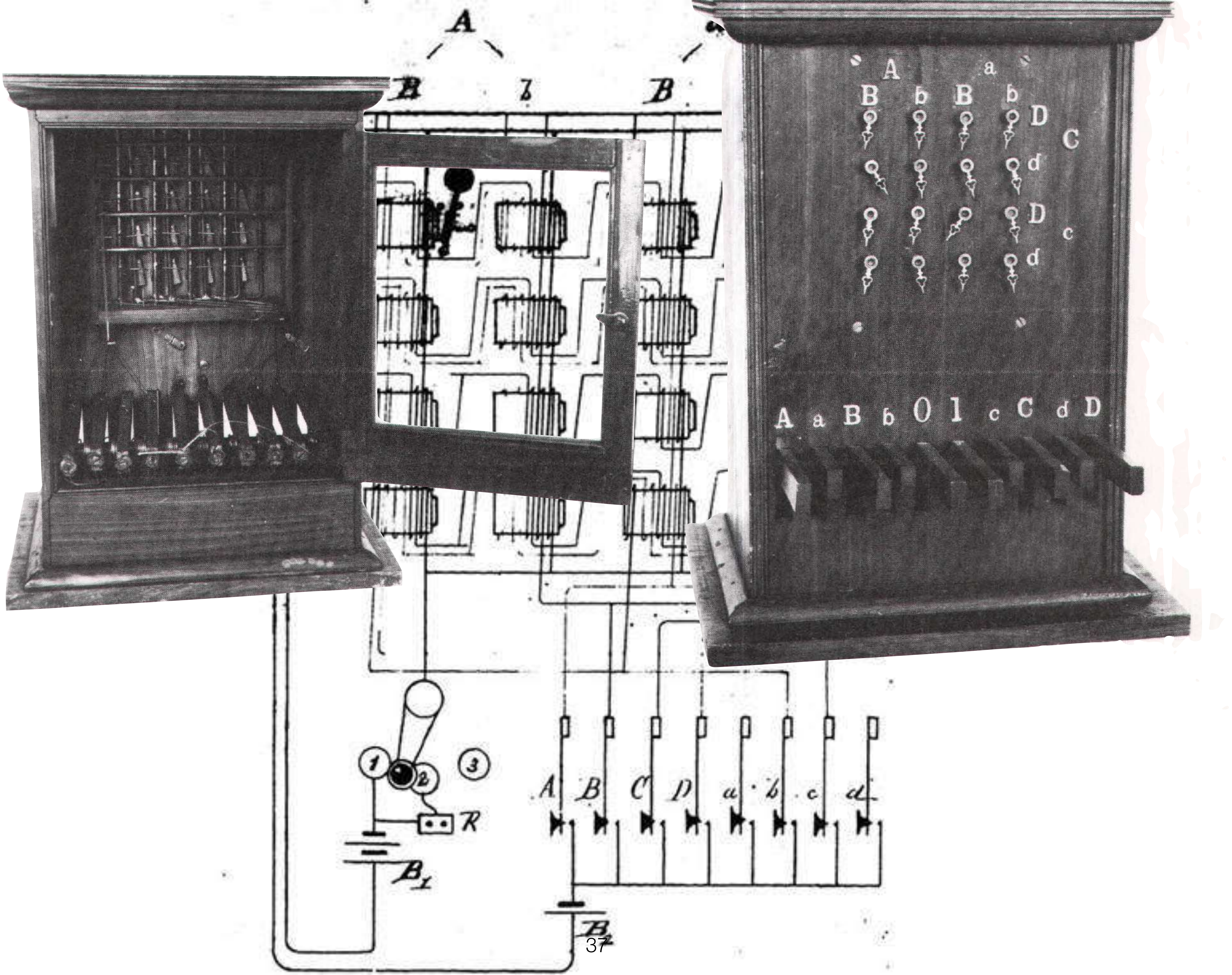
When a group of nerves are stimulated, the ganglions with which the group is most intimately connected on the whole are thrown into an active state, which in turn usually occasions movements of the body. …

*–Charles Sanders Peirce, On the Algebra of Logic, 1880*


C
c
D
d
Conjunction "Or" (Subject)
Full Stop.
ABCD
Fig. 1.
Finis.
D
c
C
b
B
a
A
Copulæ "Is"
Scale 1/6


Finis.
Full Stop.
OR
d
D
c
C
b


II.


A
B b B
A a B b B b D C d D c d
A a B b 0 1 c C d D
1 2 3
R
A B C D a b c d

Some one wrote to my husband to say that in reading an old treatise by Leibnitz … he had come upon the same formula which the Cambridge people call “Boole’s Equation.” My husband looked up Leibnitz and found his equation there and was perfectly delighted! He felt as if Leibnitz had come and shaken hands with him across the centuries. Afterwards, one of my husband’s admirers and would-be “followers” tried to persuade me that Leibnitz did not understand as much, or mean as much, as Boole had done.

*–Mary Everest Boole, 1905*

*Portrait by Renee Bolinger*

The Congress was a turning point in my intellectual life, because I met there Peano… In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. I therefore got him to give me all his works, and as soon as the Congress was over I retired to Fernhurst to study quietly every word written by him and his disciples. It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new and powerful technique for the work I had long wanted to do.

*–Bertrand Russell, 1900*

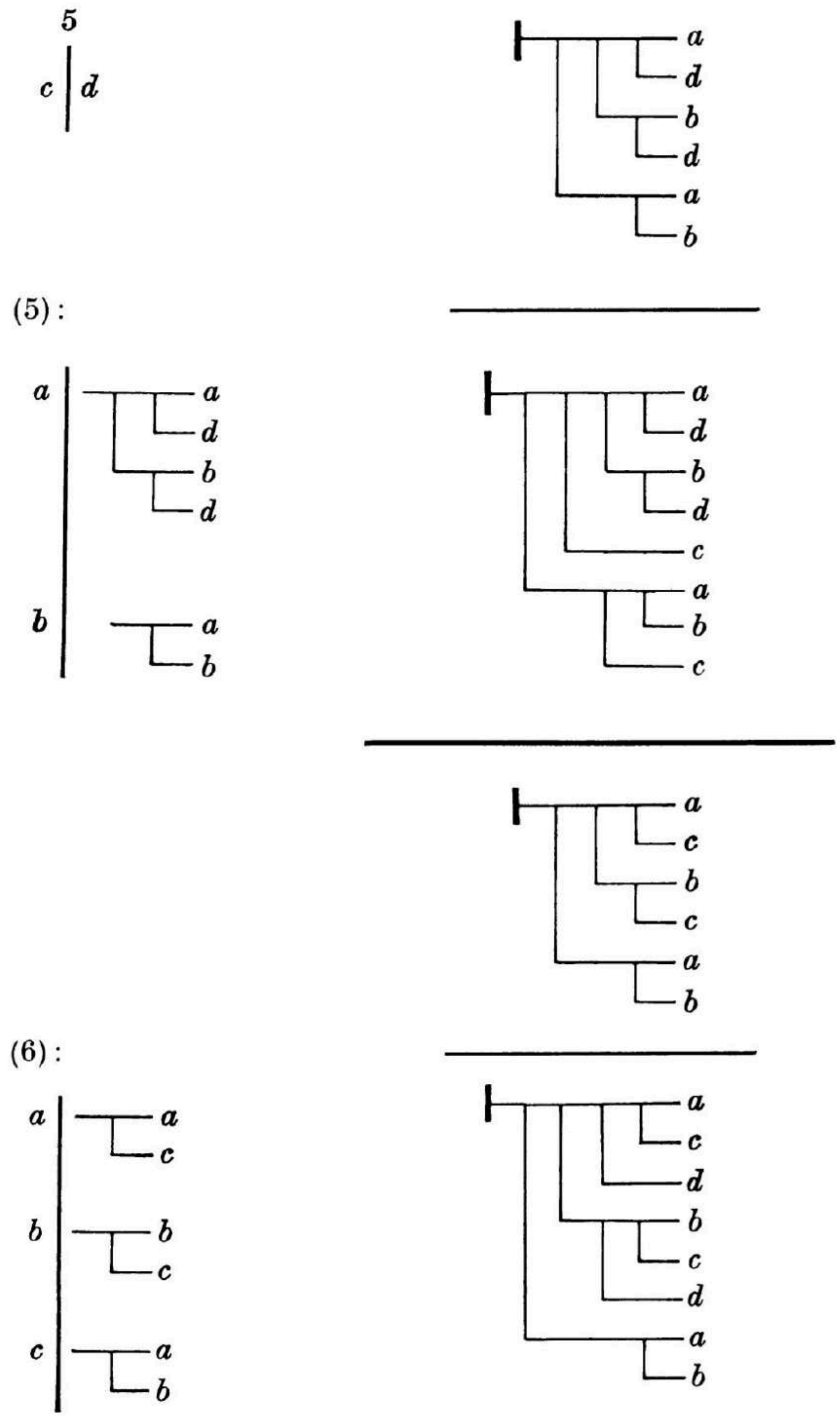
The most reliable way of carrying out a proof, obviously, is to follow pure logic, a way that, disregarding the particular characteristics of objects, depends solely on those laws upon which all knowledge rests.

… [I] had to ascertain how far one could proceed in arithmetic by means of inferences alone, with the sole support of those laws of thought that transcend all particulars… I found the inadequacy of language to be an obstacle… This deficiency led me to the idea of the present ideography. Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated.

*–Gottlob Frege,*
Begriffsschrift*, a formula language modeled upon that of arithmetic, for pure thought*, *1879*

Leibniz, too, recognized—and perhaps overrated—the advantages of an adequate system of notation. His idea of a universal characteristic, of a *calculus philosophicus* or *ratiocinator*, was so gigantic that the attempt to realize it could not go beyond the bare preliminaries. The enthusiasm that seized its originator when he contemplated the immense increase in the intellectual power of mankind that a system of notation directly appropriate to objects themselves would bring about led him to underestimate the difficulties that stand in the way of such an enterprise. But, even if this worthy goal cannot be reached in one leap, we need not despair of a slow, step-by-step approximation.

*–Gottlob Frege,*
Begriffsschrift*, a formula language modeled upon that of arithmetic, for pure thought*, *1879*




*–Gottlob Frege,*
Begriffsschrift*, a formula language modeled upon that of arithmetic, for pure thought*, *1879*

The present work promises to advance toward Leibniz's ideal of a universal language, which is still very far from its realization despite the great importance laid upon it by that brilliant philosopher!

… Frege's title, Conceptual Notation, promises too much… Instead of leaning toward a universal characteristic, the present work… definitely leans toward Leibniz's "calculus ratiocinator".

*–Ernst Schröder, Review of Frege's Begriffschrift, 1880*

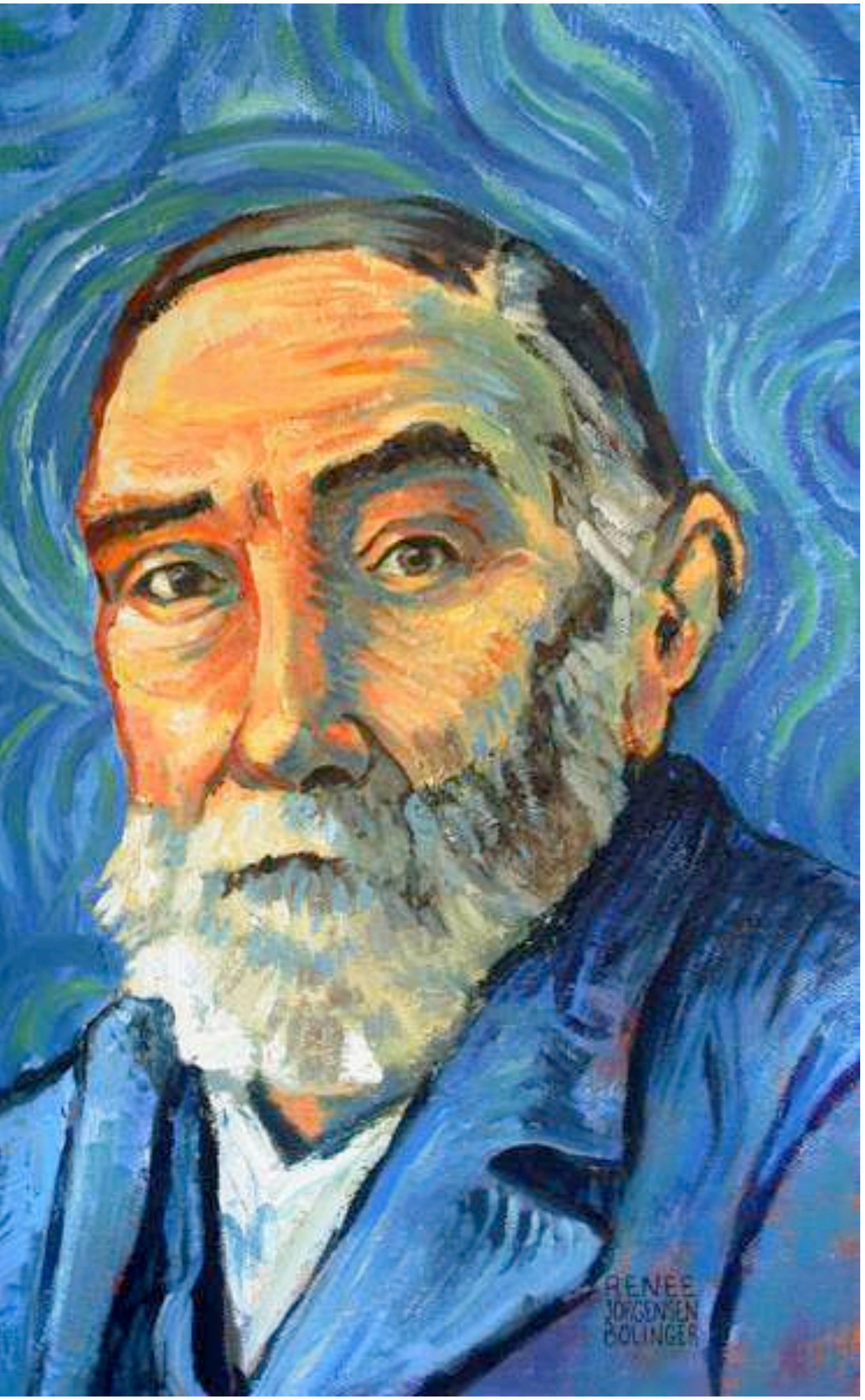
Above all, though, that reproach overlooks the fact that my purpose was quite other than Boole's. I was not trying to present an abstract logic in formulas; I was trying to express contents in an exacter and more perspicuous manner than is possible in words, by using written symbols. I was trying, in fact, to create a *'lingua characteristica'* in the Leibnizian sense, not a mere *'calculus ratiocinator'*—not that I do not recognize such a deductive calculus as a necessary constituent of a Begriffsschrift.

… Everything thus far is already to be found, with only superficial divergences, in Leibniz—of whose works relevant to this subject Boole knew nothing.

*–Frege, On the Purpose of the Begriffsschrift, 1882*

Schröder proceeds everywhere in his criticism from a direct comparableness of the Begriffsschrift with the Leibniz-Boole formula-language — a comparableness which is not to be had. His most effective contribution… is his observation that the two systems of notation are not essentially different, since it is possible to translate from one into the other. But this proves nothing. If the same subject-matter can be presented in two systems of symbols it follows automatically that translation or transcription from one to the other is possible.

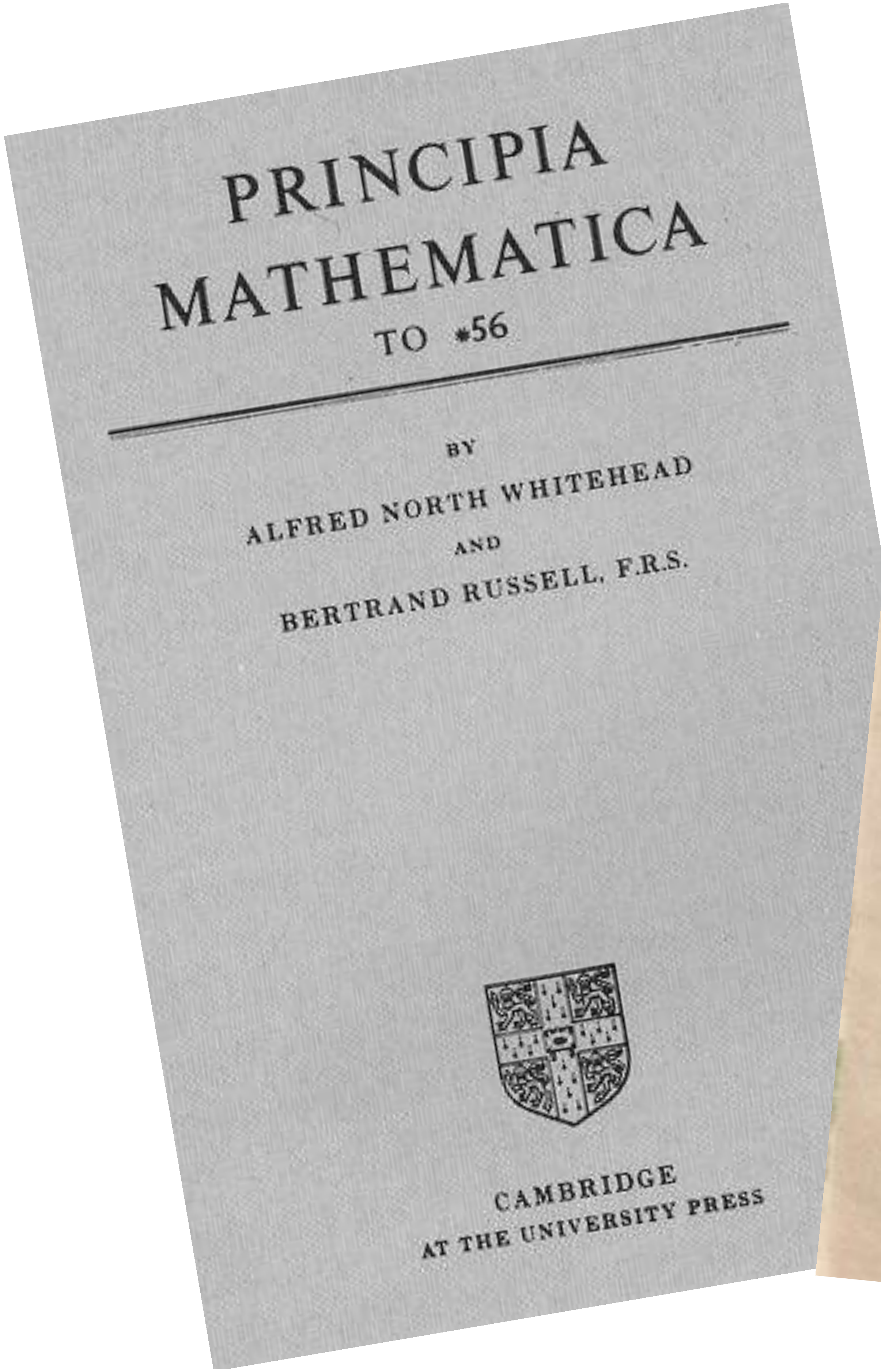
*–Frege, On the Purpose of the Begriffsschrift, 1882*




*Portrait by Renee Bolinger*

Everyone who uses words or mathematical symbols makes the claim that they mean something, and no one will expect any sense to emerge from empty symbols. But it is possible for a mathematician to perform quite lengthy calculations without understanding by his symbols anything intuitable, or with which we could be sensibly acquainted. And that does not mean that the symbols have no sense; we still distinguish between the symbols themselves and their content, even though it may be that the content can only be grasped by their aid.

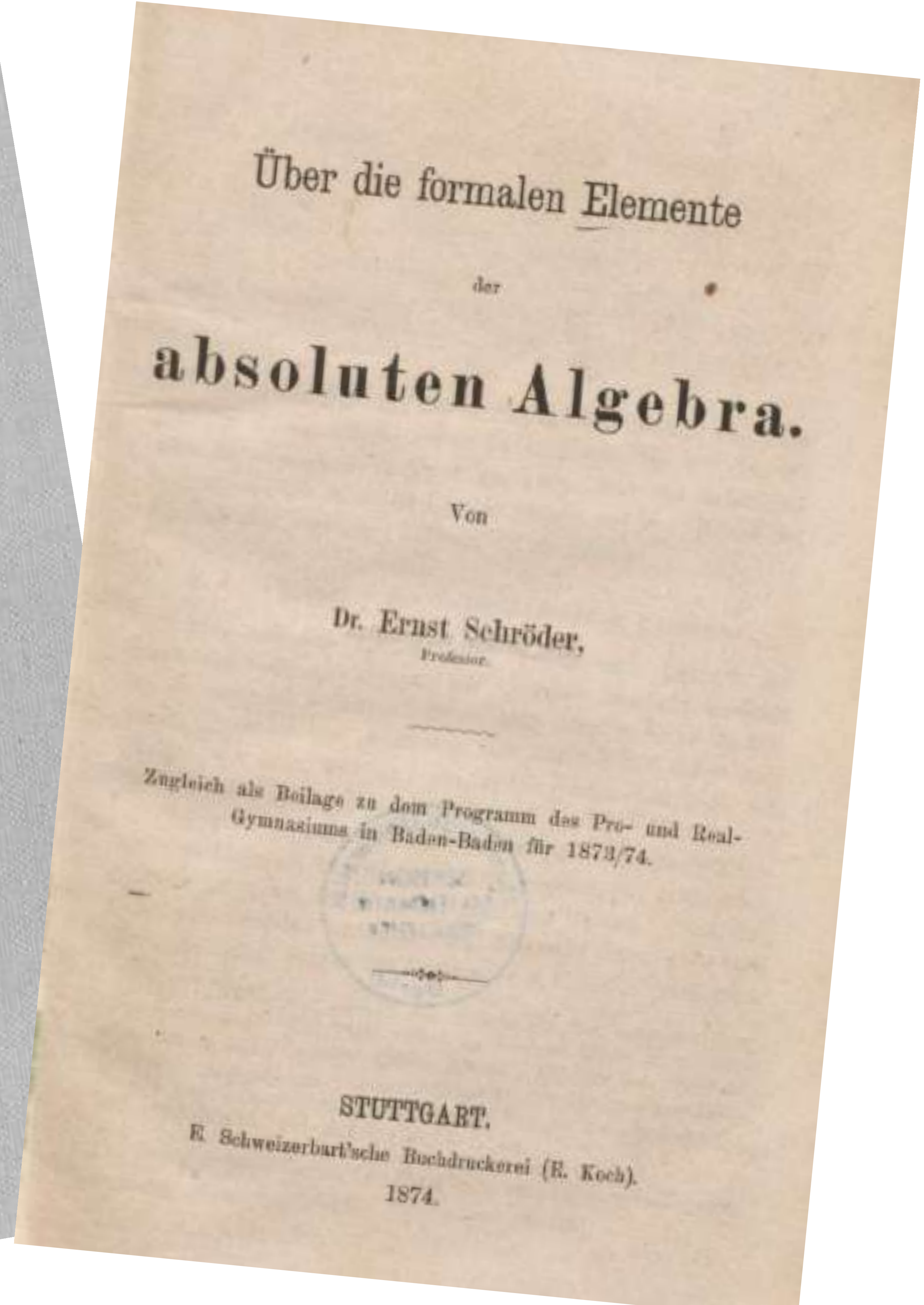
*–Frege, The Foundation of Arithmetic, 1884*



PRINCIPIA
MATHEMATICA
TO *56

BY
ALFRED NORTH WHITEHEAD
AND
BERTRAND RUSSELL, F.R.S.

CAMBRIDGE
AT THE UNIVERSITY PRESS



Über die formalen Elemente
der
absoluten Algebra.

Von
Dr. Ernst Schröder,
Professor.

Zugleich als Beilage zu dem Programm des Pro- und Real-Gymnasiums in Baden-Baden für 1873/74.

STUTTGART.
E. Schweizerbart'sche Buchdruckerei (E. Koch).
1874.

The purpose of the present paper is to propose a definition of effective calculability which is thought to correspond satisfactorily to the somewhat vague intuitive notion in terms of which problems of this class are often stated, and to show, by means of an example, that not every problem of this class is solvable.

… This definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion.

*–Alonzo Church,*
*An Unsolvable Problem of Elementary Number Theory, 1936*

We do not attach any character of uniqueness or absolute truth to any particular system of logic. The entities of formal logic are abstractions, invented because of their use in describing and systematizing facts of experience or observation, and their properties, determined in rough outline by this intended use, depend for their exact character on the arbitrary choice of the inventor.

*–Alonzo Church,*
*A Set of Postulates for the Foundation of Logic, 1932*

If this interpretation or some similar one is not allowed, it is difficult to see how the notion of an algorithm can be given any exact meaning at all.

*–Alonzo Church,*
*An Unsolvable Problem of Elementary Number Theory, 1936*

# Ratiocination

**Meaning** (concepts)

Expressed by ↓

**Language**

Manipulated by ↓

**Computation**

# Ratiocination

Computation

Language

Meaning

Computing is normally done by writing certain symbols on paper. … the number of symbols which may be printed is finite. If we were to allow an infinity of symbols, then there would be symbols differing to an arbitrarily small extent. The behaviour of the computer at any moment is determined by the symbols which he is observing, and his "state of mind" at that moment. We may suppose that there is a bound… to the number of symbols or squares which the computer can observe at one moment… We will also suppose that the number of states of mind which need be taken into account is finite. The reasons for this are of the same character as those which restrict the number of symbols. If we admitted an infinity of states of mind, some of them will be "arbitrarily close" and will be confused.

*–Alan Turing, On Computable Numbers, 1936*

We suppose… that the computation is carried out on a tape; but we avoid introducing the “state of mind” by considering a more physical and definite counterpart of it. It is always possible for the computer to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions… explaining how the work is to be continued. This note is the counterpart of the “state of mind”. We will suppose that the computer works in such a desultory manner that he never does more than one step at a sitting. The note of instructions must enable him to carry out one step and write the next note. Thus the state of progress of the computation at any stage is completely determined by the note of instructions and the symbols on the tape.

*–Alan Turing, On Computable Numbers, 1936*

In all other cases treated previously… one has been able to define them only relative to a given language, and for each individual language it is clear that the one thus obtained is not the one looked for. For the concept of computability, however, … the situation is different. By a kind of miracle it is not necessary to distinguish orders, and the diagonal procedure does not lead outside the defined notion.

*–Kurt Gödel, 1942*

[D]ue to A. M. Turing's work, a precise and unquestionably adequate definition of the general concept of formal system can now be given

*–Kurt Gödel, 1964*

# Questions?

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